MULTIPLE SOLUTIONS FOR A HENON-LIKE EQUATION ON THE ANNULUS

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Abstract. For the equation $-\Delta u = ||x|-2|^\alpha u^{p-1}$, $1<|x|<3$, we prove the existence of two solutions for α large, and of two additional solutions when p is close to the critical Sobolev exponent $2^* = 2N/(N-2)$. A symmetry–breaking phenomenon appears, showing that the least–energy solutions cannot be radial functions.

1. Introduction

In this paper we will consider the following problem:

$$\begin{cases} -\Delta u = \Psi_\alpha u^{p-1} & \text{in } \Omega, \\ u>0 & \text{in } \Omega, \\ u=0 & \text{on } \partial\Omega, \end{cases} \tag{1}$$

where $\Omega = \{x \in \mathbb{R}^N \mid 1<|x|<3\}$ is an annulus in $\mathbb{R}^N$, $N \geq 3$, $\alpha>0$, $p>2$ and Ψ_α is the radial function

$$\Psi_\alpha(x) = ||x|-2|^\alpha .$$

This equation can be seen as a natural extension to the annular domain Ω of the celebrated Hénon equation with Dirichlet boundary conditions (see [9, 11])

$$\begin{cases} -\Delta u = |x|^\alpha |u|^{p-1} & \text{for } |x|<1 \\ u=0 & \text{if } |x|=1. \end{cases} \tag{2}$$

Actually, the weight function Ψ_α reproduces on Ω a similar qualitative behavior of $|\cdot|^\alpha$ on the unit ball B of $\mathbb{R}^N$.

A standard compactness argument shows that the infimum

$$\inf_{\substack{u\in H_0^1(B)\\ u\neq 0}} \frac{\int_B |\nabla u|^2 \, dx}{\left(\int_B |x|^\alpha |u|^p \, dx\right)^{2/p}} \tag{3}$$

is achieved for any $2<p<2^*$ and any $\alpha>0$. In 1982, Ni proved in [11] that the infimum

$$\inf_{\substack{u\in H_{0,\mathrm{rad}}^1(B)\\ u\neq 0}} \frac{\int_B |\nabla u|^2 \, dx}{\left(\int_B |x|^\alpha |u|^p \, dx\right)^{2/p}} \tag{4}$$

is achieved for any $p \in (2, 2^* + \frac{2\alpha}{N-2})$ by a function in $H^1_{0,\mathrm{rad}}(B)$, the space of radial $H^1_0(B)$ functions. Thus, radial solutions of (2) exist also for (Sobolev) supercritical exponents p. Actually, radial H^1_0 elements show a power–like decay away from the origin (as a consequence of the Strauss Lemma, see [1, 18]) that combines with the

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weight $|x|^\alpha$ and provides the compactness of the embedding $H^1_{0,\mathrm{rad}}(B) \subset L^p(B)$ for any $2 < p < 2^* + \frac{2\alpha}{N-2}$.

A natural question is whether any minimizer of (3) must be radially symmetric in the range $2 < p < 2^*$ and $\alpha > 0$. Since the weight $|\cdot|^\alpha$ is an increasing function, neither rearrangement arguments nor the moving plane techniques of [8] can be applied.

Reverting to the case $\alpha > 0$, Smets *et al.* proved in [16] some symmetry–breaking results for (2). They proved that minimizers of (3) (the so-called *ground-state solutions*, or least energy solutions) cannot be radial, at least for α large enough. As a consequence, (2) has at least two solutions when α is large (see also [17]).

Later on, Serra proved in [15] the existence of at least one non–radial solution to (2) in the critical case $p = 2^*$, and in [2] the authors proved the existence of more than one solution to the same equation also for some supercritical values of p. These solutions are non-radial and they are obtained by minimization under suitable symmetry constraints.

Quite recently, Cao and Peng proved in [7] that, for p sufficiently close to 2^*, the ground-state solutions of (2) possess a unique maximum point whose distance from ∂B tends to zero as $p \to 2^*$.

This kind of result was improved in [13], where multibump solutions for the Hénon equation with almost critical Sobolev exponent p are found, by means of a finite–dimensional reduction. These solution are not radial, though they are invariant under the action of suitable subgroups of $O(N)$, and they concentrate at boundary points of the unit ball of $\mathbb{R}^N$ as $p \to 2^*$. The rôle of α is however a static one. (For more results for $p \approx 2^*$ see also [14]).

In this paper we will prove that similar phenomena take place for problem (1) on the annulus Ω. In Section 2, we present some estimates for the least energy radial solutions of (1) when $p < 2^*$ is kept fixed but $\alpha \to +\infty$. These will lead us to a first symmetry–breaking result, stating that for α sufficiently large there exist at least two solutions of (1): a global minimizer of the associated Rayleigh quotient, and a global minimizer *among radial functions*.

In Section 3, another symmetry–breaking is proved, with α fixed and $p \to 2^*$. To show this phenomenon, we will use a decomposition lemma in the spirit of P.L. Lions' concentration and compactness theory ([10]), and inspired by [7]. It will turn out that global minimizers of the same Rayleigh quotient concentrate as $p \to 2^*$ at precisely one point of the boundary $\partial\Omega$, which has two connected components. A second nonradial solution can then be found in a tricky but natural way, by minimization over functions that are "heavier" on the opposite connected component of $\partial\Omega$.

In Section 4, a third nonradial solution is singled out, by means of a linking argument. Roughly speaking, the previous nonradial solutions can be used to build a mountain pass level. In particular, this third solution will not be a local minimizer of the Rayleigh quotient.

Section 5 describes the behavior of ground-state solutions of (1) as $\alpha \to +\infty$ and $p < 2^*$ is kept fixed. Although the conclusion is not as precise as in the case $p \to 2^*$, we can nevertheless show that a sort of concentration near the boundary $\partial\Omega$ still appears.

We would like to stress that the existence of non-radial solutions in the annulus in the almost critical case $p \approx 2^*$ is not by now a surprise. When the weight disappears, i.e. $\alpha = 0$, Brezis and Nirenberg proved in [3] that the ground state solution of $-\Delta u = u^p$ in H^1_0 is not a radial function. Indeed, the authors proved that both a radial and a non-radial (positive) solution arise as $p \approx 2^*$. Their simple continuation argument can be adapted to cover our weighted equation. Subgroups

of $O(N)$ are used in [4] for the equation $-\Delta u+u=f(u)$, and some refined properties of symmetric solutions are proved. We refer to the bibliography of that paper for more references.

For more results about asymptotic estimates for solutions of the Hénon equation with α large, see [5, 6].

2. Symmetry breaking for α large

Let $H^1_{0,\mathrm{rad}}(\Omega)$ be the space of radially symmetric functions of $H^1_0(\Omega)$. With a slight but common abuse of notation, we will systematically write $u(x)=u(|x|)$ for $u \in H^1_{0,\mathrm{rad}}(\Omega)$.

Consider the minimization problem

$$S^{\mathrm{rad}}_{\alpha,p} = \inf_{u \in H^1_{0,\mathrm{rad}}(\Omega)\setminus\{0\}} R_{\alpha,p}(u), \tag{5}$$

where

$$R_{\alpha,p}(u) = \frac{\int_\Omega |\nabla u|^2\,dx}{\left(\int_\Omega \Psi_\alpha |u|^p\,dx\right)^{\frac{2}{p}}}, \qquad u \in H^1_0(\Omega)\setminus\{0\} \tag{6}$$

is the *Rayleigh quotient* associated to (1). It is known that any minimizers of (5) can be scaled so as to become solutions of (1). Therefore, we will use freely this fact in the sequel.

Remark 1. Unlike the result of [11], the fact that the annulus Ω does not contain the origin implies the existence of a radial solution of (1) for any $p>2$. Indeed, the embedding $H^1_{0,\mathrm{rad}}(\Omega) \subset L^q(\Omega)$ is compact for every $q \geq 1$, and therefore the infimum (5) is achieved by a (radial) function.

In the next Proposition, we provide an estimate of the *energy* $S^{\mathrm{rad}}_{\alpha,p}$ as $\alpha \to \infty$.

Proposition 2. *Let $p>2$. As $\alpha \to \infty$, there exist two constants C_1 and C_2 depending on p such that*

$$0 < C_1 \leq \frac{S^{\mathrm{rad}}_{\alpha,p}}{\alpha^{1+2/p}} \leq C_2 < +\infty. \tag{7}$$

Moreover, for any $M>2$ it is possible to choose the constants C_1 and C_2 independent of $p \in (2,M]$.

Proof. The upper bound C_2 can be obtain exactly as in [16]: we fix a positive, radial function $\psi \in C^\infty_0(\Omega)$, and set $\psi_\alpha(x)=\psi_\alpha(|x|)=\psi(\alpha(|x|-3+3/\alpha))$. Then

$$\begin{aligned}
\int_\Omega |\nabla \psi_\alpha|^2\,dx &= \omega_{N-1}\int_{3-\frac{2}{\alpha}}^3 \left(\psi_\alpha'(r)\right)^2 r^{N-1}\,dr \\
&= \omega_{N-1}\int_1^3 \alpha^2 \psi'(s)^2 \left(\frac{s}{\alpha}+3-\frac{3}{\alpha}\right)^{N-1} \alpha^{-1}\,ds \\
&= \alpha\,\omega_{N-1}\int_1^3 \psi'(s)^2\left(\frac{s+3\alpha-3}{\alpha s}\right)^{N-1} s^{N-1}\,ds \\
&\leq 3^{N-1}\alpha\int_\Omega |\nabla\psi|^2\,dx, \quad \left(\text{since } 1 \leq \tfrac{s+3\alpha-3}{\alpha s} \leq 3\right)
\end{aligned}$$

and

$$\int_\Omega \Psi_\alpha \psi_\alpha^p\,dx \geq \left(1-\frac{2}{\alpha}\right)^\alpha \alpha^{-1}\int_\Omega \psi^p\,dx.$$

This proves that $S^{\mathrm{rad}}_{\alpha,p} \le C(\alpha,p)\alpha^{1+\frac{2}{p}}$, where

$$C(\alpha,p) = 3^{N-1}\frac{\int_\Omega |\nabla\psi|^2 dx}{(1-\frac{2}{\alpha})^{\frac{2\alpha}{p}}(\int_\Omega \psi^p(x)\,dx)^{\frac{2}{p}}} \le C_2 \quad \text{for any } p>2 \text{ and } \alpha>1.$$

To find the lower bound C_1, we will perform some scaling. Let us define the functions $\psi_1\colon [1,2]\to[1,2]$ and $\psi_2\colon [2,3]\to[2,3]$ as follows:

$$\psi_1(r) = 2-(2-r)^\beta, \quad \psi_2(r) = 2+(r-2)^\beta, \tag{8}$$

where $\beta\in(0,1)$ will be chosen hereafter. It is clear that we can obtain a piecewise C^1 homeomorphism $\psi\colon[1,3]\to[1,3]$ by gluing ψ_1 and ψ_2. Now, for any radial function $u\in H^1_0(\Omega)$, we set $v(\rho)=u(\psi(\rho))$ and compute:

$$\begin{aligned}
\int_\Omega |\nabla u|^2\,dx &= \omega_{N-1}\int_1^3 |u'(r)|^2 r^{N-1}\,dr \\
&\ge \omega_{N-1}\int_1^3 |u'(r)|^2\,dr \\
&= \omega_{N-1}\left(\int_1^2 |v'(\rho)|^2\frac{1}{\psi_1'(\rho)}\,d\rho + \int_2^3 |v'(\rho)|^2\frac{1}{\psi_2'(\rho)}\,d\rho\right) \\
&= \omega_{N-1}\frac{1}{\beta}\left(\int_1^2 |v'(\rho)|^2(2-\rho)^{1-\beta}\,d\rho \right.\\
&\quad \left. + \int_2^3 |v'(\rho)|^2(\rho-2)^{1-\beta}\,d\rho\right) \\
&= \omega_{N-1}\frac{1}{\beta}\int_1^3 |v'(\rho)|^2|\rho-2|^{1-\beta}\,d\rho \\
&\ge \omega_{N-1}\frac{1}{\beta}\int_1^3 |v'(\rho)|^2|\rho-2|\,d\rho. \qquad (9)
\end{aligned}$$

Choosing $\beta = 1/(\alpha+1)$,

$$\begin{aligned}
\int_\Omega \Psi_\alpha(x)|u(|x|)|^p\,dx &= \omega_{N-1}\int_1^3 \Psi_\alpha(r)|u(r)|^p r^{N-1}\,dr \\
&\le 3^{N-1}\omega_{N-1}\int_1^3 \Psi_\alpha(r)|u(r)|^p\,dr \\
&= 3^{N-1}\omega_{N-1}\left(\int_1^2 \Psi_\alpha(\psi_1(\rho))|v(\rho)|^p\psi_1'(\rho)\,d\rho\right.\\
&\quad \left. + \int_2^3 \Psi_\alpha(\psi_2(\rho))|v(\rho)|^p\psi_2'(\rho)\,d\rho\right) \\
&= 3^{N-1}\omega_{N-1}\beta\int_1^3 |v(\rho)|^p\,d\rho. \qquad (10)
\end{aligned}$$

Since we are integrating over Ω and $0\notin\Omega$, the integral $\int_{1/2}^1 |v'(\rho)|^2\rho^{N-1}\,d\rho$ is finite if and only if $\int_{1/2}^1 |v'(\rho)|^2\,d\rho$ is finite. Therefore,

$$R_{\alpha,p}(u) \ge C\alpha^{1+\frac{2}{p}} \inf_{\substack{v\in H^1_0(\Omega)\\ v\ne 0}} \frac{\int_1^3 |v'(\rho)|^2|4\rho-3|\,d\rho}{\left(\int_1^3 |v(\rho)|^p\,d\rho\right)^{2/p}}. \tag{11}$$

where C depends only on N. To complete the proof, we will show that the right-hand side of (11) is greater than zero. This follows from some general Hardy–type

inequality (see [12], Theorem 1.14), but we present here an elementary proof for the sake of completeness. Indeed, given $v \in H^1_{0,\mathrm{rad}}(\Omega)$, we can write for $\rho \in [1,2]$

$$\begin{aligned}
|v(\rho)| &= |v(\rho)-v(1)| \leq \int_1^\rho |v'(t)||2-t|^{1/2}\frac{dt}{|2-t|^{1/2}} \\
&\leq \left(\int_1^\rho |v'(t)|^2|2-t|\,dt\right)^{1/2}\left(\int_1^\rho \frac{dt}{|2-t|}\right)^{1/2} \\
&\leq \left(\int_1^3 |v'(t)|^2|2-t|\,dt\right)^{1/2}(-\log|2-\rho|)^{1/2}.
\end{aligned}$$

Hence

$$\begin{aligned}
\int_1^2 |v(\rho)|^p\,d\rho &\leq \left(\int_1^3 |v'(\rho)|^2|2-\rho|\,d\rho\right)^{p/2}\int_1^2 (-\log(2-\rho))^{p/2}\,d\rho \\
&= \left(\int_1^3 |v'(\rho)|^2|2-\rho|\,d\rho\right)^{p/2}\int_0^\infty t^{p/2}e^{-t}\,dt \\
&= \Gamma\left(\frac{p+2}{2}\right)\left(\int_1^3 |v'(\rho)|^2|2-\rho|\,d\rho\right)^{p/2},
\end{aligned}$$

and in a similar way

$$\int_2^3 |v(\rho)|^p\,d\rho \leq \Gamma\left(\frac{p+2}{2}\right)\left(\int_1^3 |v'(\rho)|^2|2-\rho|\,d\rho\right)^{p/2}.$$

Therefore

$$\int_1^3 |v'(\rho)|^2|2-\rho|\,d\rho \geq \frac{1}{2^{2/p}\Gamma\left(\frac{p+2}{2}\right)^{2/p}}\left(\int_1^3 |v(\rho)|^p\,d\rho\right)^{\frac{2}{p}}.$$

This implies that the infimum in (11) is strictly positive and for any $M > 2$ there exists a constant $C_1 > 0$ such that $2^{-2/p} \geq C_1\,\Gamma\left(\frac{p+2}{2}\right)^{2/p}$ for any $p \in (2,M]$, since the Gamma function is positive, C^∞ and $\Gamma\left(\frac{p+2}{2}\right) \sim (p/2)^{p/2}\mathrm{e}^{-p/2}\sqrt{\pi p}$ for $p \to +\infty$. We finally collect (9) and (10) to get the desired conclusion

$$S^{\mathrm{rad}}_{\alpha,p} \geq C_1\,\alpha^{1+\frac{2}{p}}.$$

□

Set now

$$S_{\alpha,p} = \inf_{\substack{u\in H^1_0(\Omega)\\ u\neq 0}} R_{\alpha,p}(u). \tag{12}$$

It is easily proved that for p subcritical $S_{\alpha,p}$ is attained by a function $u_{\alpha,p}$ that satisfies (up to a scaling) equation (1).

In order to prove that any solution $u_{\alpha,p}$ is not radial (at least for α large) we need an estimate from above of the level $S_{\alpha,p}$.

Lemma 3. *Let $p \in (2,2^*)$. There exists $\bar\alpha$ such that for $\alpha \geq \bar\alpha$*

$$S_{\alpha,p} \leq C\alpha^{2-N+\frac{2N}{p}}. \tag{13}$$

Proof. The proof essentially follows the same techniques of [16].

Let ψ be a positive smooth function with support in the unit ball B. Let us consider $\psi_\alpha(x) = \psi(\alpha(x-x_\alpha))$, where $x_\alpha = (3-\frac{1}{\alpha},0,\dots,0)$. Since ψ_α has support in the ball $B(x_\alpha,\frac{1}{\alpha})$, by the change of variable $y = \alpha(x-x_\alpha)$ we obtain:

$$\int_\varnothing \Psi_\alpha(x)\psi_\alpha^p(x)\,dx = \int_{B(x_\alpha,\frac{1}{\alpha})} ||x|-2|^\alpha\psi_\alpha^p(x)\,dx \geq \left(1-\frac{2}{\alpha}\right)^\alpha\alpha^{-N}\int_B \psi^p(y)\,dy$$

Moreover

$$\int_{\varnothing} |\nabla \psi_\alpha|^2 \, dx = \alpha^2 \int_{\varnothing} |\nabla \psi(\alpha(x - x_\alpha))|^2 \, dx = \alpha^{2-N} \int_B |\nabla \psi|^2 \, dx,$$

so that

$$S_{\alpha,p} \leq R_{\alpha,p}(\psi_\alpha) \leq C\alpha^{2-N+\frac{2N}{p}}$$

for all α sufficiently large. This proves the Lemma. □

By comparing (13) and (7), we deduce a first symmetry–breaking result.

Theorem 4. *Let $p \in (2, 2^*)$. For α sufficiently large, any ground state $u_{\alpha,p}$ is a non-radial function.*

Proof. From (13) and (7) it follows that $S_{\alpha,p} < S^{\mathrm{rad}}_{\alpha,p}$ when α is large. □

3. Symmetry breaking as $p \to 2^*$

In this section we consider α fixed, p close to 2^* and we establish the following

Theorem 5. *Let $\alpha > 0$. For p close to 2^* the quotient $R_{\alpha,p}$ has at least two non radial local minima.*

We briefly explain how the proof proceeds. Of course, we already know that any global minimizer of $R_{\alpha,p}$ produces a first solution $u_{\alpha,p}$. As the Theorem 6 states, this solution concentrates at precisely one point of the boundary $\partial\Omega$. Since this boundary has two connected components, we will minimize $R_{\alpha,p}$ over the set Λ of H^1_0 functions which “concentrate” at the opposite component of the boundary. A careful estimate is proved in order to show that minimizers fall inside the interior of Λ.

Consider now any minimizer $u_{\alpha,p}$. As in [7] we have a description of the profile of $u_{\alpha,p}$ as $p \to 2^*$.

Theorem 6. *Let $p \in (2, 2^*)$ and $\alpha > 0$. Any minimum $u_{\alpha,p}$ of $R_{\alpha,p}(u)$ in $H^1_0 \setminus \{0\}$ satisfies (passing to a subsequence) for some $x_0 \in \partial\Omega$*

i) $|\nabla u_{\alpha,p}|^2 \to \mu\delta_{x_0}$ *weakly in the sense of measure as $p \to 2^*$,*
ii) $|u_{\alpha,p}|^{2^*} \to \nu\delta_{x_0}$ *weakly in the sense of measure as $p \to 2^*$,*

where $\mu > 0$ and $\nu > 0$ are such that $\mu \geq S_{0,2^*}\nu^{2/2^*}$ and δ_x is the Dirac mass at x.

Proof. This result can be proven by using, with suitable modifications, the same arguments of [7]. □

To get a second local minimizer, we will assume without loss of generality that any $u_{\alpha,p}$ concentrates at some point on the sphere $|x| = 3$ (a similar argument holds if $u_{\alpha,p}$ concentrates at some point x with $|x| = 1$). After a rotation, we can even assume that any $u_{\alpha,p}$ concentrates at the point $(3, 0, \dots, 0)$.

Let

$$\Omega^- = \left\{x \in \mathbb{R}^N \mid 1 < |x| < 2\right\}, \quad \Omega^+ = \left\{x \in \mathbb{R}^N \mid 2 < |x| < 3\right\}$$

and

$$\Sigma = \left\{ u \in H^1_0 \setminus \{0\} \mid \int_{\Omega^+} |\nabla u|^2 \, dx = \int_{\Omega^-} |\nabla u|^2 \, dx \right\}.$$

Let us denote

$$T_{\alpha,p} = \inf_{u \in \Sigma} R_{\alpha,p}(u).$$

We have the following uniform estimate for $T_{\alpha,p}$.

Proposition 7. *Let $\alpha > 0$. There exists $\delta > 0$ such that*

$$\liminf_{p\to 2^*} T_{\alpha,p} > S_{0,2^*} + \delta.$$

Proof. We first prove that $T_{\alpha,p}$ is achieved by a function $v_{\alpha,p} \in \Sigma$. Consider a minimizing sequence $\{u_n\}$ for $T_{\alpha,p}$. We can exploit the homogeneity of $R_{\alpha,p}$ and assume that $\int_\Omega |\nabla u_n|^2\, dx = 1$. Up to a subsequence, u_n converges to $v = v_{\alpha,p}$ weakly in $H^1_0(\Omega)$ and strongly in $L^q(\Omega)$, for all $q \in (2, 2^*)$. All we have to check is that $v \in \Sigma$ (proving *a posteriori* that the convergence of u_n to v is strong). From the strong convergence in $L^q(\Omega)$ we have that

$$R_{\alpha,p}(v) \le \frac{1}{\left(\int_\Omega \Psi_\alpha(x)|v|^p dx\right)^{2/p}} = T_{\alpha,p} \tag{14}$$

and in particular $v \neq 0$. It is not restrictive to suppose that $v \ge 0$ in Ω. For the sake of contraddiction, assume that

$$\int_{\Omega^+} |\nabla v|^2 < \frac{1}{2}.$$

Fix a nonnegative smooth function $\varphi_1 \in C_0^\infty(\Omega^+)$, $\varphi_1 \neq 0$ and $\delta \ge 0$. Setting $u = v + \delta\varphi_1$ from the positivity of v and φ_1, we have, for $\delta > 0$,

$$\int_\Omega \Psi_\alpha(x)|v|^p dx < \int_\Omega \Psi_\alpha(x)|u|^p dx. \tag{15}$$

Now,

$$\int_{\Omega^+} |\nabla u|^2 dx = \int_{\Omega^+} |\nabla v|^2 dx + 2\delta \int_{\Omega^+} \nabla v \cdot \nabla\varphi_1 dx + \delta^2 \int_{\Omega^+} |\nabla \varphi_1|^2 dx$$

If we define $g_1 : [0, +\infty) \to \mathbb{R}$ by

$$g_1(\delta) = \int_{\Omega^+} |\nabla v|^2 dx + 2\delta \int_{\Omega^+} \nabla v \cdot \nabla\varphi_1 dx + \delta^2 \int_{\Omega^+} |\nabla \varphi_1|^2 dx$$

we see that g_1 is continuous and $g_1(0) < \frac{1}{2}$, $\lim\limits_{\delta\to+\infty} g_1(\delta) = +\infty$. Hence there exists $\delta_1 > 0$ with $g_1(\delta_1) = \frac{1}{2}$. We can reason in an analogous way if $\int_{\Omega^-} |\nabla v|^2 dx < 1/2$ in order to find $\delta_2 \ge 0$ and $\varphi_2 \ge 0$ such that $\int_{\Omega^-} |\nabla(v + \delta_2\varphi_2)|^2 dx = \frac{1}{2}$.

From (15), this shows that there exists $w = v + \delta_1\varphi_1 + \delta_2\varphi_2 \in \Sigma$ such that

$$R_{\alpha,p}(w) < T_{\alpha,p}$$

which gives a contraddiction. Finally we must have that $v_{\alpha,p} \in \Sigma$, is a minimum point.

Moreover for any $\alpha > 0$, and $2 < p < 2^*$ we have $T_{\alpha,p} \ge S_{\alpha,p}$. We want to prove that the inequality is strict at least for $p \to 2^*$. Indeed assume on the contrary that

$$\liminf_{p\to 2^*} T_{\alpha,p} = \liminf_{p\to 2^*} R_{\alpha,p}(v_{\alpha,p}) = S_{0,2^*}.$$

From the definition of $S_{0,2^*}$ and Hölder inequality we get, for a subsequence $p = p_k \to 2^*$

$$\begin{aligned} S_{0,2^*} &\le \frac{\int_\Omega |\nabla v_{\alpha,p}|^2\ dx}{\left(\int_\Omega |v_{\alpha,p}|^{2^*}\ dx\right)^{2/2^*}} \le |\Omega|^{\frac{(2^*-p)2}{2^*p}} \frac{\int_\Omega |\nabla v_{\alpha,p}|^2\ dx}{(\int_\Omega |v_{\alpha,p}|^p\ dx)^{2/p}} \\ &\le |\Omega|^{\frac{(2^*-p)2}{2^*p}} \frac{\int |\nabla v_{\alpha,p}|^2\ dx}{(\int_\Omega \Psi_\alpha(x)|v_{\alpha,p}|^p\ dx)^{2/p}} = S_{0,2^*} + o(1) \end{aligned} \tag{16}$$

since the weight satisfies $\Psi_\alpha(x) \le 1$. In particular

$$\frac{\int_\Omega |\nabla v_{\alpha,p}|^2 \ dx}{\left(\int_\Omega |v_{\alpha,p}|^{2^*} \ dx\right)^{2/2^*}} \to S_{0,2^*},$$

and $v_{\alpha,p}$ is a minimizing sequence of $S_{0,2^*}$.

In the same way as Cao and Peng did in [7], Theorem 1.1, we can prove that $v_{\alpha,p}$ concentrates at precisely one point one of the boundary $\partial\Omega$. This contradicts the fact that $\int_{\Omega^+} |\nabla v_{\alpha,p}|^2 \ dx = \int_{\Omega^-} |\nabla v_{\alpha,p}|^2 \ dx$. □

Consider now the points

$$x_{0,\varepsilon} = x_0 = \left(3 - \frac{1}{|\log\varepsilon|}, 0, \dots, 0\right), \quad x_{1,\varepsilon} = x_1 = \left(1 + \frac{1}{|\log\varepsilon|}, 0, \dots, 0\right)$$

and

$$U(x) = \frac{1}{(1+|x|)^{(N-2)/2}}.$$

We recall that $S_{0,2^*}$ is not achieved on any proper subset of $\mathbb{R}^N$, and that it is independent of Ω. However, it is known that $S_{0,2^*}(\mathbb{R}^N)$ is achieved, and all the minimizers can be written in the form

$$\mathcal{U}_{\theta,y}(x) = \frac{1}{(\theta^2 + |x-y|^2)^{\frac{N-2}{2}}}, \quad \theta > 0,\ y \in \mathbb{R}^N.$$

We set

$$U_\varepsilon^i(x) = \varepsilon^{-\frac{N-2}{2}}\ U\left(\frac{x - x_i}{\sqrt{\varepsilon}}\right) = \frac{1}{(\varepsilon + |x-x_i|^2)^{\frac{N-2}{2}}},$$

and denote by φ_i $(i = 0,1)$ two *cut-off* functions such that $0 \le \varphi_i \le 1$, $|\nabla\varphi_i| \le C|\log\varepsilon|$ for some constant $C > 0$, and

$$\varphi_i(x) = \begin{cases} 1, & \text{if } |x - x_i| < \frac{1}{2|\log\varepsilon|} \\ 0, & \text{if } |x - x_i| \ge \frac{1}{|\log\varepsilon|}. \end{cases}$$

The following Lemma shows that the truncated functions

$$u_\varepsilon^i(x) = \varphi_i(x) U_\varepsilon^i(x), \quad i = 0,1, \tag{17}$$

are almost minimizers for $S_{0,2^*}$. We omit the proof of this fact, since it is an easy modification of the argument of Cao and Peng in [7].

Lemma 8. *Let $\alpha > 0$. There results*

$$\lim_{p \to 2^*} R_{\alpha,p}(u_\varepsilon^i) = S_{0,2^*} + K(\varepsilon)$$

with $\lim\limits_{\varepsilon\to 0} K(\varepsilon) = 0$.

Remark 9. A direct consequence of Lemma 8 is that $S_{0,2^*} = S_{\alpha,2^*}$. Indeed $S_{0,2^*} \le S_{\alpha,2^*}$ since $\Psi_\alpha(|x|) \le 1$. On the other hand by Lemma 8

$$R_{\alpha,2^*}(u_\varepsilon^i) = \lim_{p\to 2^*} R_{\alpha,p}(u_\varepsilon^i) = S_{0,2^*} + K(\varepsilon).$$

Therefore $S_{0,2^*} + K(\varepsilon) \ge S_{\alpha,2^*}$ for every $\varepsilon > 0$. Letting $\varepsilon \to 0$ we conclude $S_{0,2^*} \ge S_{\alpha,2^*}$.

We are now ready to prove the Theorem 5.

Proof of Theorem 5. Let $u_{\alpha,p}$ be a ground state solution. Let us suppose that it concentrates on the outer boundary. Consider the open subset

$$\Lambda = \left\{ u \in H_0^1(\Omega) : \int_{\Omega^-} |\nabla u|^2 \, dx > \int_{\Omega^+} |\nabla u|^2 \, dx \right\}.$$

The infimum of $R_{\alpha,p}$ on $\overline{\Lambda}$ is achieved. However it cannot be achieved on the boundary $\partial\Lambda = \Sigma$. Indeed, by Proposition 7,

$$\inf_{\Sigma} R_{\alpha,p} > S_{0,2^*} + \delta \text{ as } p \to 2^*$$

and

$$\inf_{\Lambda} R_{\alpha,p}(u) \leq R_{\alpha,p}(u_\varepsilon^1) \to S_{0,2^*} + K_1(\varepsilon) \text{ as } p \to 2^*$$

since $u_\varepsilon^1 \in \Lambda$ for ε small enough. Then the infimum is achieved in a interior point of Λ and is therefore a critical point of $R_{\alpha,p}$.

□

Remark 10. Theorem 6 shows that any ground state solution "concentrates" in a single point at the boundary as $p \to 2^*$ and consequently this solution is not radial. This symmetry breaking can be also proved by using a continuation argument as in [3]. Indeed, (16) shows that $\lim_{p\to 2^*} S_{\alpha,p} = S_{0,2^*}$, and since $S_{0,2^*} < S_{0,2^*}^{\mathrm{rad}}$ we conclude as in [3] that ground states of $S_{\alpha,p}$ cannot be radially symmetric as $p \to 2^*$.

4. EXISTENCE OF A THIRD NON-RADIAL SOLUTION

In the previous section we proved the existence of two solutions of (1) which are local minima of the Rayleigh quotient for p near 2^*. One would expect another critical point of $R_{\alpha,p}$ located in some sense between these minimum points. This is precisely the idea we are going to pursue further in the current section.

For ε small enough let $u_\varepsilon^i = \varphi_i U_\varepsilon^i$, $i \in \{0,1\}$, be defined as in (17). We will prove that $R_{\alpha,p}$ has the Mountain Pass geometry.

Let us introduce the *mountain–pass level*

$$\beta = \beta(\alpha, p) = \inf_{\gamma\in\Gamma} \max_{t\in[0,1]} R_{\alpha,p}(\gamma(t)),$$

where $\Gamma = \{\gamma \in C([0,1], H_0^1(\Omega)) \mid \gamma(0) = u_\varepsilon^0,\ \gamma(1) = u_\varepsilon^1\}$ is the set of continuous paths joining u_ε^0 with u_ε^1. We claim that β is a critical value for $R_{\alpha,p}$.

We begin to prove that β is larger, uniformly with respect to ε, than the values of the functional $R_{\alpha,p}$ at the points u_ε^0 and u_ε^1.

Lemma 11. *Let $M_\varepsilon = \max\{R_{\alpha,p}(u_\varepsilon^0), R_{\alpha,p}(u_\varepsilon^1)\}$. There exists $\sigma > 0$ such that $\beta \geq M_\varepsilon + \sigma$ uniformly with respect to ε.*

Proof. We prove that there exists σ such that for all $\gamma \in \Gamma$

$$\max R_{\alpha,p}(\gamma(t)) \geq M_\varepsilon + \sigma.$$

A simple continuity argument shows that for every $\gamma \in \Gamma$ there exists t_γ such that $\gamma(t_\gamma) \in \Sigma$, where

$$\Sigma = \left\{ u \in H_0^1 \setminus \{0\} \mid \int_{\Omega^+} |\nabla u|^2 \, dx = \int_{\Omega^-} |\nabla u|^2 \, dx \right\}.$$

Indeed the map $t \in [0,1] \mapsto \int_{\Omega^+} |\nabla\gamma(t)|^2 \, dx - \int_{\Omega^-} |\nabla\gamma(t)|^2 \, dx$ is continuous and it takes a negative value at $t = 0$ and a positive value at $t = 1$. Now Proposition 7 implies, for p near 2^* the existence of $\delta > 0$ with

$$\max_{t\in[0,1]} R_{\alpha,p}(\gamma(t)) \geq R_{\alpha,p}(\gamma(t_\gamma)) \geq \inf_{u\in\Sigma} R_{\alpha,p}(u) \geq S_{0,2^*} + \delta.$$

On the other hand, for ε sufficiently small,

$$M_\varepsilon < S_{0,2^*} + \frac{\delta}{2}.$$

This concludes the proof. $\square$

The previous estimate allows us to show that β is a critical level for $R_{\alpha,p}$. Therefore a further nonradial solution to (1) arises.

Proposition 12. *There exist $\bar\alpha > 0$ and $2 < \bar p < 2^*$ such that for all $\alpha \geq \bar\alpha$ and $\bar p \leq p < 2^*$ it results that β is a critical value for $R_{\alpha,p}$ and it is attained by a non-radial function.*

Proof. From the previous result we can apply a deformation argument (see [1, 19]) to prove that β is a critical level and it is attained (since the PS condition is satisfied) by a function w. From the asymptotic estimate (7) for the radial level $S^{\text{rad}}_{\alpha,p}$, one has that there exists a constant C independent from p such that

$$S^{\text{rad}}_{\alpha,p} \geq C\alpha^{1+2/p}.$$

In particular

$$S^{\text{rad}}_{\alpha,p} \to +\infty \quad \text{as } \alpha \to +\infty.$$

This allows us to choose α_0 such that $S^{\text{rad}}_{\alpha,p} \geq 3S_{0,2^*}$ for all $\alpha \geq \alpha_0$.

Define $\zeta \in \Gamma$ by $\zeta(t) = tu^1_\varepsilon + (1-t)u^0_\varepsilon$ for all $t \in [0,1]$, and let $\tau \in [0,1]$ be such that $R_{\alpha,p}(\zeta(\tau)) = \max_{t\in[0,1]} R_{\alpha,p}(\zeta(t))$.

Since u^1_ε and u^0_ε have disjoint supports one has, for ε sufficiently small,

$$\begin{aligned}
R_{\alpha,p}(w) &= \beta \leq R_{\alpha,p}(\zeta(\tau)) = \frac{\int_\Omega |\nabla(\tau u^1_\varepsilon + (1-\tau)u^0_\varepsilon)|^2\,dx}{\left(\int_\Omega \Psi_\alpha|\tau u^1_\varepsilon + (1-\tau)u^0_\varepsilon|^p\,dx\right)^{2/p}} \\
&= \frac{\int_\Omega \tau^2|\nabla u^1_\varepsilon|^2\,dx + \int_\Omega (1-\tau)^2|\nabla u^0_\varepsilon|^2\,dx}{\left(\tau^p\int_\Omega \Psi_\alpha|u^1_\varepsilon|^p\,dx + (1-\tau)^p\int_\Omega \Psi_\alpha|u^0_\varepsilon|^p\,dx\right)^{2/p}} \\
&\leq \frac{\tau^2\int_\Omega|\nabla u^1_\varepsilon|^2\,dx}{\left(\tau^p\int_\Omega \Psi_\alpha|u^1_\varepsilon|^p\,dx\right)^{2/p}} + \frac{(1-\tau)^2\int_\Omega|\nabla u^0_\varepsilon|^2\,dx}{\left((1-\tau)^p\int_\Omega \Psi_\alpha|u^0_\varepsilon|^p\,dx\right)^{2/p}} \\
&= R_{\alpha,p}(u^0_\varepsilon) + R_{\alpha,p}(u^1_\varepsilon) \leq 2M_\varepsilon < 3S_{0,2^*} \leq S^{\text{rad}}_{\alpha,p}.
\end{aligned}$$

This concludes the proof. $\square$

5. Behaviour of the ground-state solutions for α large

This section is devoted to the analysis of a ground state solution as $\alpha \to +\infty$. Even in this case this solution tends to "concentrate" at the boundary $\partial\Omega$. However, this concentration is much weaker than the concentration as $p \to 2^*$.

We use the notation $C(r_1, r_2) = \{x \in \mathbb{R}^N \mid r_1 < |x| < r_2\}$. Let δ be sufficiently small (say $\delta < \frac{1}{2}$) and ϕ be a smooth cut-off function such that $0 \leq \phi \leq 1$ with

$$\phi(x) = \begin{cases} 1, & x \in C(1, 1+\delta) \cup C(3-\delta, 3) \\ 0, & x \in C(2-\delta, 2+\delta) \end{cases} \tag{18}$$

From now on, since $p \in (2, 2^*)$ is fixed we denote a ground state solution of problem (1) $u_{\alpha,p}$ with u_α.

Proposition 13. *Let u_α be such that $R_{\alpha,p}(u_\alpha) = S_{\alpha,p}$. If ϕ is as in (18), then*

$$R_{\alpha,p}(\phi u_\alpha) = S_{\alpha,p} + o(S_{\alpha,p}) \quad \text{as } \alpha \to +\infty. \tag{19}$$

Proof. It is not restrictive, by the homogeneity of $R_{\alpha,p}$, to assume $\int_\Omega |\nabla u_\alpha|^2\,dx = 1$. We split the proof into two steps.
Step 1. We claim that

$$\int_\Omega \Psi_\alpha (u_\alpha\phi)^p\,dx = \int_\Omega \Psi_\alpha u_\alpha^p\,dx + o\left(\int_\Omega \Psi_\alpha u_\alpha^p\,dx\right) \tag{20}$$

Indeed, suppose on the contrary that

$$\limsup_{\alpha\to\infty} \frac{\int_\Omega \Psi_\alpha u_\alpha^p (1-\phi^p)\,dx}{\int_\Omega \Psi_\alpha u_\alpha^p\,dx} = \beta > 0$$

This implies that, up to some subsequence,

$$\frac{\int_\Omega \Psi_\alpha u_\alpha^p (1-\phi^p)\,dx}{\int_\Omega \Psi_\alpha u_\alpha^p\,dx} > \beta/2 > 0$$

Since $1-\phi^p \equiv 0$ on $C(1,1+\delta)\cup C(3-\delta,3)$ we have

$$\begin{aligned}\int_\Omega \Psi_\alpha u_\alpha^p(1-\phi^p)\,dx &= \int_{C(1+\delta,3-\delta)} \Psi_\alpha u_\alpha^p(1-\phi^p)\,dx\\ &\le (1-\delta)^\alpha \int_\Omega u_\alpha^p(1-\phi^p)\,dx \le (1-\delta)^\alpha \int_\Omega u_\alpha^p\,dx.\end{aligned}$$

Therefore

$$\int_\Omega u_\alpha^p\,dx \ge (1-\delta)^{-\alpha}\int_\Omega \Psi_\alpha u_\alpha^p(1-\phi^p)\,dx$$

Now

$$\frac{\int_\Omega u_\alpha^p\,dx}{\int_\Omega \Psi_\alpha u_\alpha^p\,dx} \ge (1-\delta)^{-\alpha}\frac{\int_\Omega \Psi_\alpha u_\alpha^p(1-\phi^p)\,dx}{\int_\Omega \Psi_\alpha u_\alpha^p\,dx} \ge (1-\delta)^{-\alpha}\frac{\beta}{2}.$$

Since $S_{\alpha,p}^{p/2} = \left(\int_\Omega \Psi_\alpha u_\alpha^p\,dx\right)^{-1}$ the last inequality can be written as

$$S_{\alpha,p}^{p/2} \ge \frac{\beta}{2}\,\frac{(1-\delta)^{-\alpha}}{\int_\Omega u_\alpha^p\,dx} \ge \frac{\beta}{2}(1-\delta)^{-\alpha} S_{0,p}^{p/2},$$

where

$$S_{0,p} = \inf_{u\neq 0}\frac{\int_\Omega |\nabla u|^2\,dx}{(\int_\Omega u^p\,dx)^{2/p}}$$

On the other hand from (13) one has the estimate

$$S_{\alpha,p}^{p/2} \le C\alpha^{p-\frac{N}{2}p+N},$$

which gives a contradiction for α large. This proves (20).
Step 2. Now we prove that

$$\int_\Omega |\nabla u_\alpha\phi|^2\,dx = \int_\Omega |\nabla u_\alpha|^2\,dx + o(1) = 1 + o(1). \tag{21}$$

It is not difficult to prove that u_α satisfies the problem

$$\begin{cases} -\Delta u_\alpha = S_{\alpha,p}^{p/2}\Psi_\alpha u_\alpha^{p-1} & \text{in } \Omega,\\ u_\alpha > 0 & \text{in } \Omega,\\ u_\alpha = 0 & \text{on } \partial\Omega,\end{cases} \tag{22}$$

Since $\|\nabla u_\alpha\|_2 = 1$, up to subsequences, we have that, as $\alpha\to\infty$,

$$u_\alpha \to u \quad \text{weakly in } H_0^1(\Omega), \text{ strongly in } L^q(\Omega), \text{ and a.e.}$$

We now prove that $u = 0$. Indeed, multiplying equation (22) by a smooth function ψ with $\operatorname{supp}\psi \subset\subset \emptyset$ and integrate, we obtain

$$\int_\Omega \nabla u_\alpha \nabla \psi \, dx = \int_\Omega S_{\alpha,p}^{p/2} \Psi_\alpha u_\alpha^{p-1} \psi \, dx \to 0, \quad \alpha \to +\infty$$

since, by (13), $S_{\alpha,p}^{p/2}\Psi_\alpha \to 0$ uniformly on $\operatorname{supp}\psi$ and u_α is uniformly bounded in L^q for $1 \leq q < 2^*$. Hence $\int_\Omega \nabla u \cdot \nabla \varphi \, dx = 0$ for all $\varphi \in C_0^\infty(\Omega)$. Since $u \in H_0^1(\Omega)$, this implies that $u = 0$.

Now we estimate the difference

$$\left| \int_\Omega |\nabla u_\alpha|^2 \, dx - \int_\Omega |\nabla (u_\alpha \phi)|^2 \, dx \right| \leq \tag{23}$$
$$\leq \int_\Omega |\nabla u_\alpha|^2 (1-\phi^2) \, dx + \int_\Omega |\nabla \phi|^2 u_\alpha^2 \, dx + 2 \left| \int_\Omega \nabla u_\alpha \nabla \phi u_\alpha \phi \, dx \right|$$

The last terms tend to zero thanks to the strong convergence in L^q for all $q \in [1, 2^*)$. In order to estimate the term $\int_\Omega |\nabla u_\alpha|^2 (1-\phi^2) \, dx$, we multiply (22) by $u_\alpha(1-\phi^2) = u_\alpha \eta$ and integrate. Since $\operatorname{supp}\eta = \operatorname{supp}(1-\phi^2) \subset\subset \Omega$ we have

$$\int_\Omega \nabla u_\alpha \nabla (\eta u_\alpha) \, dx = \int_\Omega S_{\alpha,p}^{p/2} \Psi_\alpha u_\alpha^p \eta \, dx,$$

so that

$$\begin{aligned} \left| \int_\Omega |\nabla u_\alpha|^2 \eta \, dx \right| &\leq \left| \int_\Omega u_\alpha \nabla \eta \nabla u_\alpha \, dx \right| + \left| \int_\Omega S_{\alpha,p}^{p/2} \Psi_\alpha u_\alpha^p \eta \, dx \right| \\ &\leq ||\nabla \eta||_\infty \int_{\operatorname{supp}\eta} |\nabla u_\alpha u_\alpha| \, dx + \left| \int_{\operatorname{supp}\eta} S_{\alpha,p}^{p/2} \Psi_\alpha u_\alpha^p \eta \, dx \right| \to 0. \end{aligned}$$

□

In proposition 13 we proved that the infimum of the Rayleigh quotient $R_{\alpha,p}$ is essentially attained by the function ϕu_α. Thanks to the definition of ϕ, we can decompose $\phi u_\alpha = u_{\alpha,1} + u_{\alpha,2}$, where $u_{\alpha,1}$ vanishes in $C(2-\delta, 3)$ and $u_{\alpha,2}$ vanishes in $C(1, 2+\delta)$. The following proposition is the main step in order to prove that the function ϕu_α concentrates at the boundary.

Proposition 14. *Let $\phi u_\alpha = u_{\alpha,1} + u_{\alpha,2}$, where $\operatorname{supp} u_{\alpha,1} \subset C(1, 2-\delta)$ and $\operatorname{supp} u_{\alpha,2} \subset C(2+\delta, 3)$, and $\lambda_\alpha = \int_\Omega \Psi_\alpha u_{\alpha,1}^p \, dx / \int_\Omega \Psi_\alpha u_{\alpha,2}^p \, dx$. If $\lim_{n\to\infty} \lambda_{\alpha_n} = L$ for a sequence $\alpha_n \to \infty$ then either $L = 0$ or $L = +\infty$.*

Remark 15. For the quantity $\lambda_\alpha = \int_\Omega \Psi_\alpha u_{\alpha,1}^p \, dx / \int_\Omega \Psi_\alpha u_{\alpha,2}^p \, dx$, we cannot exclude the case $\limsup_{\alpha\to+\infty} \lambda_\alpha = +\infty$ and $\liminf_{\alpha\to+\infty} \lambda_\alpha = 0$. If a *uniqueness* result for the minimizer u_α were known, then it would be easy to conclude that $\alpha \mapsto u_\alpha$ is continuous. Therefore λ_α would be continuous, too, and we could replace both the lower and the upper limit by a unique limit. In general, one does not expect such a uniqueness property for any p and any α. However, when $p \approx 2^*$ we suspect that the uniqueness argument of [14] may be applied to our setting.

Proof. By the definition of $u_{\alpha,1}$ and $u_{\alpha,2}$ we have

$$R_{\alpha,p}(\phi u_\alpha) = \frac{\int_\Omega |\nabla u_{\alpha,1}|^2 \, dx + \int_\Omega |\nabla u_{\alpha,2}|^2 \, dx}{\left(\int_\Omega \Psi_\alpha u_{\alpha,1}^p dx + \int_\Omega \Psi_\alpha u_{\alpha,2}^p dx \right)^{\frac{2}{p}}}. \tag{24}$$

Since u_α is a positive solution, we can say that $\lambda_\alpha > 0$. We obtain the following identity:

$$
\begin{aligned}
R_{\alpha,p}(\phi u_\alpha) &= \frac{\int_\Omega |\nabla u_{\alpha,1}|^2\,dx + \int_\Omega |\nabla u_{\alpha,2}|^2\,dx}{\left(\lambda_\alpha \int_\Omega \Psi_\alpha u_{\alpha,2}^p\,dx + \int_\Omega \Psi_\alpha u_{\alpha,2}^p\,dx\right)^{2/p}} \\
&= \frac{\int_\Omega |\nabla u_{\alpha,1}|^2\,dx}{(\lambda_\alpha+1)^{2/p}\left(\int_\Omega \Psi_\alpha u_{\alpha,2}^p\,dx\right)^{2/p}} + \frac{\int_\Omega |\nabla u_{\alpha,2}|^2\,dx}{(\lambda_\alpha+1)^{2/p}\left(\int_\Omega \Psi_\alpha u_{\alpha,2}^p\,dx\right)^{2/p}} \\
&= \frac{\lambda_\alpha^{\frac{2}{p}}\int_\Omega |\nabla u_{\alpha,1}|^2\,dx}{(\lambda_\alpha+1)^{\frac{2}{p}}\left(\int_\Omega \Psi_\alpha u_{\alpha,1}^p\,dx\right)^{2/p}} + \frac{\int_\Omega |\nabla u_{\alpha,2}|^2\,dx}{(\lambda_\alpha+1)^{\frac{2}{p}}\left(\int_\Omega \Psi_\alpha u_{\alpha,2}^p\,dx\right)^{2/p}}. \qquad (25)
\end{aligned}
$$

By the definition of $S_{\alpha,p}$ each quotient $R_{\alpha,p}(u_{\alpha,1})$ and $R_{\alpha,p}(u_{\alpha,2})$ in the last term is greater than or equal to $S_{\alpha,p}$. Therefore by Proposition 13 and equation (25) one obtains

$$S_{\alpha,p} + o(S_{\alpha,p}) \geq \frac{1+\lambda_\alpha^{\frac{2}{p}}}{(\lambda_\alpha+1)^{\frac{2}{p}}} S_{\alpha,p}. \qquad (26)$$

We notice that the function $f(x) = \frac{1+x^{2/p}}{(x+1)^{2/p}}$ is strictly greater than 1 for every $x > 0$, $f(0) = 1$ and $f(x) \to 1$ as $x \to +\infty$. Moreover it is increasing in $[0,1]$ and decreasing in $[1,+\infty)$ and $\max_{x>0} f(x) = f(1) = 2^{1-2/p}$. Let $L \in \Lambda$ and $\{\alpha_n\}$ a sequence such that $\lambda_{\alpha_n} \to L$ as $n \to +\infty$. Passing to the limit in (26), we obtain that $1 \geq \frac{1+L^{2/p}}{(L+1)^{2/p}}$ and so either $L = +\infty$, or $L = 0$. □

Corollary 16. *With the notation of Proposition 14, for any sequence $\{\alpha_n\}$ such that $\lambda_{\alpha_n} \to 0$ one has*

$$\lim_{n\to+\infty} \frac{\int_\Omega |\nabla u_{\alpha_n,1}|^2\,dx}{\int_\Omega |\nabla u_{\alpha_n,2}|^2\,dx} = 0. \qquad (27)$$

Proof. Let

$$\xi_\alpha = \frac{\int_\Omega |\nabla u_{\alpha,1}|^2\,dx}{\int_\Omega |\nabla u_{\alpha,2}|^2\,dx}$$

and suppose that $\limsup_{n\to\infty} \xi_{\alpha_n} > 0$. Up to subsequences, $\xi_{\alpha_n} > \xi > 0$ for some ξ. Therefore we have

$$
\begin{aligned}
S_{\alpha_n,p} + o(S_{\alpha_n,p}) &= \frac{\int_\Omega |\nabla u_{\alpha_n,1}|^2\ dx + \int_\Omega |\nabla u_{\alpha_n,2}|^2\ dx}{\left(\int_\Omega \Psi_{\alpha_n} u_{\alpha_n,1}^p dx + \int_\Omega \Psi_{\alpha_n} u_{\alpha_n,2}^p dx\right)^{\frac{2}{p}}} \\
&= \frac{(1+\xi_{\alpha_n})\int_\Omega |\nabla u_{\alpha_n,2}|^2\ dx}{\left(\int_\Omega \Psi_{\alpha_n} u_{\alpha_n,2}^p dx\right)^{\frac{2}{p}}(1+\lambda_{\alpha_n})} \\
&\geq R_{\alpha_n,p}(u_{\alpha_n,2})\frac{1+\xi}{1+o(1)} \geq (1+\xi)S_{\alpha_n,p} + o(S_{\alpha_n,p}),
\end{aligned}
$$

which is a contradiction. Hence

$$\xi_{\alpha_n} = \frac{\int_\Omega |\nabla u_{\alpha_n,1}|^2\,dx}{\int_\Omega |\nabla u_{\alpha_n,2}|^2\,dx} \to 0.$$

□

Remark 17. An immediate consequence of the previous results is that in particular for any α_n such that $\lambda_{\alpha_n} \to 0$

$$\lim_{n\to+\infty} \int_\Omega |\nabla u_{\alpha_n,1}|^2\,dx = 0. \qquad (28)$$

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